

# FAITHFULNESS OF DIRECTED COMPLETE POSETS BASED ON SCOTT CLOSED SET LATTICES

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**ABSTRACT.** By Thron, a topological space  $X$  has the property that  $C(X)$  isomorphic to  $C(Y)$  implies  $X$  is homeomorphic to  $Y$  iff  $X$  is sober and  $T_D$ , where  $C(X)$  and  $C(Y)$  denote the lattices of closed sets of  $X$  and  $T_0$  space  $Y$ , respectively. When we consider dcpos (directed complete posets) equipped their Scott topologies, a similar question arises: which dcpos  $P$  have the property that for any dcpo  $Q$ ,  $C_\sigma(P)$  isomorphic to  $C_\sigma(Q)$  implies  $P$  is isomorphic to  $Q$  (such a dcpo  $P$  will be called Scott closed set lattice faithful, or SCL-faithful in short)? Here  $C_\sigma(P)$  and  $C_\sigma(Q)$  denote the lattices of Scott closed sets of  $P$  and  $Q$ , respectively. Following a characterization of continuous (quasicontinuous) dcpos in terms of  $C_\sigma(P)$ , one easily deduces that every continuous (quasicontinuous) dcpo is SCL-faithful. Note that the Scott space of every continuous (quasicontinuous) dcpo is sober. Compared with Thron's result, one naturally asks whether every SCL-faithful dcpo is sober (with the Scott topology). In this paper we shall prove that some classes of dcpos are SCL-faithful, these classes contain some dcpos whose Scott topologies are not bounded sober. These results will help to obtain a complete characterization of SCL-faithful dcpos in the future.

## 1. INTRODUCTION

In [12], Thron proved the interesting result: a topological space  $X$  has the property that  $C(X)$  isomorphic to  $C(Y)$  implies  $X$  is homeomorphic to  $Y$  iff  $X$  is sober and  $T_D$ , where  $C(X)$  and  $C(Y)$  denote the lattices of closed sets of  $X$  and  $T_0$  space  $Y$ , respectively. A directed complete poset (dcpo, for short)  $P$  will be called Scott closed set lattice faithful, or SCL-faithful in short if for any dcpo  $Q$ ,  $P$  is isomorphic to  $Q$  whenever the Scott-closed-set lattice  $C_\sigma(P)$  of  $P$  and  $C_\sigma(Q)$  of  $Q$  are isomorphic. One of the classic result in domain theory is that a dcpo  $P$  is continuous iff the lattice  $C_\sigma(P)$  is a completely distributive lattice (Theorem II-1.14 of [3]). From this it follows that every continuous dcpo is SCL-faithful. In a similar way, one deduces that every quasicontinuous dcpo is SCL-faithful. Compared with Thron's result, one naturally asks whether every SCL-faithful dcpo is sober in their Scott topology.

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In [9], Johnstone constructed the first dcpo whose Scott topology is not sober. Later Isbell [8] constructed a complete lattice whose Scott topology is not sober and Kou [10] gave a dcpo whose Scott topology is well-filtered but not sober. In this paper, we will prove that some classes of dcpos, including all quasicontinuous dcpos as well as Johnstone's and Kou's examples, are SCL-faithful. The full characterization of all SCL-faithful dcpos is still open.

## 2. PRELIMINARIES

For any subset  $A$  of a poset  $P$ , let  $\uparrow A = \{x \in P : y \leq x \text{ for some } y \in A\}$  and  $\downarrow A = \{x \in P : x \leq y \text{ for some } y \in A\}$ . A subset  $A$  is called an upper set if  $A = \uparrow A$ , and a lower set if  $A = \downarrow A$ . A subset  $U$  of a poset  $P$  is Scott open if (i)  $U = \uparrow U$  and (ii) for any directed subset  $D$ ,  $\bigvee D \in U$  implies  $D \cap U \neq \emptyset$ , whenever  $\bigvee D$  exists. All Scott open sets of a poset  $P$  form a topology on  $P$ , denoted by  $\sigma(P)$  and called the Scott topology on  $P$ . The complements of Scott open sets are called Scott closed sets. Clearly, a subset  $A$  is Scott closed iff (i)  $A = \downarrow A$  and (ii) for any directed subset  $D \subseteq A$ ,  $\bigvee D \in A$  whenever  $\bigvee D$  exists. The set of all Scott closed sets of  $P$  will be denoted by  $C_\sigma(P)$ . The space  $(P, \sigma(P))$  is denoted by  $\Sigma P$ , called the Scott space of  $P$  (See [3] for more about Scott spaces).

A poset  $P$  is directed complete if its every directed subset has a supremum. A directed complete poset is briefly called a dcpo.

A subset  $A$  of a topological space is irreducible if  $A \subseteq F_1 \cup F_2$  with  $F_1$  and  $F_2$  closed, then  $A \subseteq F_1$  or  $A \subseteq F_2$  holds. The set of all nonempty irreducible closed subsets of space  $X$  will be denoted by  $\text{Irr}(X)$ .

For any  $T_0$  topological space  $(X, \tau)$ , the specialization order  $\leq_\tau$  on  $X$  is defined by  $x \leq_\tau y$  iff  $x \in \text{cl}(\{y\})$  where " $\text{cl}(\cdot)$ " means taking closure.

**Remark 2.1.** (1) For any topological space  $X$ ,  $(\text{Irr}(X), \subseteq)$  is a dcpo. If  $\mathcal{D}$  is a directed subset of  $\text{Irr}(X)$ , the supremum of  $\mathcal{D}$  in  $(\text{Irr}(X), \subseteq)$  equals  $\text{cl}(\bigcup \mathcal{D})$  (the closure of  $\bigcup \mathcal{D}$ ), which is the same as the supremum of  $\mathcal{D}$  in the complete lattice of all closed sets of  $X$ .

(2) For any  $x \in X$ ,  $\text{cl}(\{x\}) \in \text{Irr}(X)$ . A  $T_0$  space  $X$  is called sober if  $\text{Irr}(X) = \{\text{cl}(\{x\}) : x \in X\}$ , that is, every nonempty irreducible closed set is the closure of a point.

(3) If  $(X, \tau)$  and  $(Y, \eta)$  are topological spaces such that the open set lattices  $(\tau, \subseteq)$  and  $(\eta, \subseteq)$  of  $X$  and  $Y$  are isomorphic, then the posets  $\text{Irr}(X)$  and  $\text{Irr}(Y)$  are isomorphic.

For a  $T_0$  space  $X$ , a sobrification of  $X$  is a sober space  $Y$  together with a continuous mapping  $\eta_X : X \rightarrow Y$ , such that for any continuous mapping  $f : X \rightarrow Z$  with  $Z$  sober, there is a unique continuous mapping  $\hat{f} : Y \rightarrow Z$  such that  $f = \hat{f} \circ \eta_X$ . The sobrification of a  $T_0$  space is unique up to homeomorphism.

**Remark 2.2.** The following facts on sobrifications are well-known.

(1) If  $Y$  is a sober space, then  $Y$  is a sobrification of a  $T_0$  space  $X$  iff the closed set lattice  $C(X)$  of  $X$  is isomorphic to the closed set lattice  $C(Y)$  of  $Y$  (Equivalently, the open set lattice of  $Y$  is isomorphic to that of  $X$ ).

(2) The set  $\text{Irr}(X)$  of all nonempty closed irreducible sets of a  $T_0$  space  $X$  equipped the hull-kernel topology is a sobrification of  $X$ , where the mapping  $\eta_X : X \rightarrow \text{Irr}(X)$  is defined by  $\eta_X(x) = \text{cl}(\{x\})$  for all  $x \in X$ . The closed sets of the hull-kernel topology consists of all sets of the form  $h(A) = \{F \in \text{Irr}(X) : F \subseteq A\}$  ( $A$  is a closed set of  $X$ ).

A  $T_0$  space will be called Scott sobrifiable if there is a dcpo  $P$  such that  $\Sigma P$  is the sobrification of  $X$ . Also for a  $T_0$  space  $(X, \tau)$ ,  $(X, \tau)$  is homeomorphic to  $\Sigma P$  for some poset  $P$  iff  $(X, \tau)$  is homeomorphic to the Scott space  $\Sigma(X, \leq_\tau)$ .

**Lemma 2.3.** *A  $T_0$  space  $(X, \tau)$  is Scott sobrifiable iff for any Scott closed set  $\mathcal{F}$  of the dcpo  $\text{Irr}(X)$ , there is a closed set  $A$  of  $X$  such that  $\mathcal{F} = h(A)$ .*

*Proof.* Note that  $\text{Irr}(X)$  equipped with the hull-kernel topology is a sobrification of  $X$ . Also every  $h(A) = \{F \in \text{Irr}(X) : F \subseteq A\}$  is a Scott closed set of the dcpo  $(\text{Irr}(X), \subseteq)$ , where  $A$  is a closed set of  $X$ . Thus the hull kernel topology on  $\text{Irr}(X)$  is contained in the Scott topology of  $\text{Irr}(X)$ . Thus  $(X, \tau)$  is Scott sobrifiable iff there is a dcpo  $P$  such that  $\text{Irr}(X)$  is homeomorphic to  $\Sigma P$ . This is then equivalent to that the hull kernel topology on  $\text{Irr}(X)$  coincides with the Scott topology on  $(\text{Irr}(X), \subseteq)$ , hence every Scott closed set of  $(\text{Irr}(X), \subseteq)$  equals  $h(A)$  for some closed set  $A$  of  $(X, \tau)$ .  $\square$

A topological space  $(X, \tau)$  is called a d-space (or monotone convergence space) if (i)  $X$  is  $T_0$ , (ii) the poset  $(X, \leq_\tau)$  is a dcpo, and (iii) for any directed subset  $D \subseteq X$ ,  $D$  converges (as a net) to  $\bigvee D$ . If  $(X, \tau)$  is a d-space, then every closed set  $F$  of  $X$  is a Scott closed set of the dcpo  $(X, \leq_\tau)$ .

**Remark 2.4.** (1) *Every sober space is a d-space.*

(2) *Every Scott space of a dcpo is a d-space.*

**Lemma 2.5.** *Let  $(X, \tau)$  be a d-space. If  $\{x_i : i \in I\}$  is a directed subset of  $(X, \leq_\tau)$ , then the supremum  $\sup\{cl(\{x_i\}) : i \in I\}$  of  $\{cl(\{x_i\}) : i \in I\}$  in  $\text{Irr}(X)$  equals  $cl(\{x\})$ , where  $x = \bigvee\{x_i : i \in I\}$ .*

### 3. MAIN RESULTS

In this section, we establish some classes of SCL-faithful dcpos, using irreducible sets, quasicontinuous elements and M property, respectively.

A  $T_0$  space is called bounded-sober if every nonempty upper bounded (with respect to the specialization order on  $X$ ) closed irreducible subset of the space is the closure of a point [13]. Every sober space is bounded-sober, the converse implication is not true.

If  $X$  is a  $T_0$  space such that every irreducible closed *proper* subset is the closure of an element, then  $X$  is bounded-sober.

In the following, a dcpo whose Scott space is sober (bounded-sober) will be simply called a sober (bounded-sober) dcpo.

**Lemma 3.1.** *For a bounded-sober dcpo  $P$ ,  $\Sigma P$  is Scott sobrifiable if and only if  $P$  is sober.*

*Proof.* We only need to check that if  $\Sigma P$  is not sober, then it is not Scott sobrifiable.

Since  $\Sigma P$  is not sober, there is a nonempty irreducible closed set  $F$  such that  $F$  is not the closure of any point. By that  $\Sigma P$  is bounded-sober, one can verify that the set  $\mathcal{F} = \downarrow_{\text{Irr}(\Sigma P)} \{cl(\{x\}) : x \in F\}$  is a Scott closed set of  $\text{Irr}(\Sigma P)$ . But any closed set  $B$  of  $\Sigma P$  containing all  $cl(\{x\})(x \in F)$  must contain  $F$ , thus  $h(B) \neq \mathcal{F}$ . By Lemma 2.3,  $\Sigma P$  is not Scott sobrifiable.  $\square$

In the following, we shall write  $P \cong Q$  if the two posets  $P$  and  $Q$  are isomorphic.

**Theorem 3.2.** *Let  $P$  be a sober dcpo. For any bounded-sober dcpo  $Q$ , if  $C_\sigma(P) \cong C_\sigma(Q)$  then  $P \cong Q$ .*

*Proof.* Let  $Q$  be a bounded-sober dcpo such that  $C_\sigma(P) \cong C_\sigma(Q)$ . Then  $\Sigma P$  is a sobrification of  $\Sigma Q$ . By Lemma 3.1,  $\Sigma Q$  is sober, therefore  $\Sigma P$  and  $\Sigma Q$  are homeomorphic, which implies  $P \cong Q$ .  $\square$

**Definition 3.3.** *An element  $a$  of a poset  $P$  is called down-linear if the subposet  $\downarrow a = \{x \in P : x \leq a\}$  is a chain (for any  $x_1, x_2 \in \downarrow a$ , it holds that either  $x_1 \leq x_2$  or  $x_2 \leq x_1$ ).*

The image of a down-linear element under an order isomorphism is clearly down-linear.

**Lemma 3.4.** *Let  $X$  be a  $d$ -space. If  $F \in \text{Irr}(X)$  is a down-linear element of the poset  $\text{Irr}(X)$ , then there exists a unique  $x \in X$  such that  $F = \text{cl}(\{x\})$ .*

*Proof.* First, the set  $\{\text{cl}(\{x\}) : x \in F\}$  is a subset of  $\downarrow F$  in  $\text{Irr}(X)$ , so it is a chain. Thus  $\{x : x \in F\}$  is a chain of  $(X, \leq_\tau)$ . Let  $x = \sup\{x : x \in F\}$ . Then noticing that  $F$  is closed, we have  $\text{cl}(\{x\}) = F$ .  $\square$

In the following, for a dcpo  $P$ , we shall use  $\text{Irr}_\sigma(P)$  to denote the dcpo of all nonempty irreducible Scott closed subsets of  $P$ . Without specification, irreducible sets of a poset mean the irreducible sets with respect to the Scott topology.

**Theorem 3.5.** *Let  $P$  be a dcpo satisfying the following conditions:*

(DL-sup) *for any nonempty irreducible Scott closed proper set  $F$ ,  $F$  is either a down-linear element of  $\text{Irr}_\sigma(P)$  or it is the supremum of a directed set of down-linear irreducible closed sets.*

*Then  $P$  is SCL-faithful.*

*Proof.* Let dcpo  $P$  satisfy the above condition (DL-sup) and  $Q$  be a dcpo such that  $C_\sigma(P) \cong C_\sigma(Q)$ .

(1) Let  $F \in \text{Irr}_\sigma(P)$  and  $F \neq P$ . If  $F$  is down-linear, then by Lemma 3.4,  $F$  is the closure of a unique point. If  $F$  is the supremum of a directed set of down-linear irreducible closed sets, then by Lemma 3.4,  $F$  can be represented as the directed supremum of  $\text{cl}(\{x_i\}) (i \in J)$ . Thus,  $\{x_i : i \in J\}$  is a directed set of  $P$ . Let  $x = \sup\{x_i : i \in J\}$ . Then noticing that  $F$  is closed, we have that  $\text{cl}(\{x\}) = F$  is also a closure of a point.

(2) Since  $C_\sigma(P) \cong C_\sigma(Q)$ ,  $Q$  also satisfies condition (DL-sup). As the proof of (1) only make use of condition (DL-sup), so every nonempty closed irreducible proper subset of  $\Sigma Q$  is the closure of a point.

By the definition of the bounded-sobriety, we see that (1) and (2) imply that  $\Sigma P$  and  $\Sigma Q$  are all bounded-sober.

(3) If either  $\Sigma P$  or  $\Sigma Q$  is sober, then by Theorem 3.2,  $P \cong Q$ . If neither  $\Sigma P$  nor  $\Sigma Q$  is sober, then  $P$  and  $Q$  are irreducible sets and are not the closure of any point. Thus  $Q \cong \{\text{cl}(\{y\}) : y \in Q\} \cong \text{Irr}_\sigma(Q) - \{Q\} \cong \text{Irr}_\sigma(P) - \{P\} \cong \{\text{cl}(\{x\}) : x \in P\} \cong P$ , as desired.  $\square$

**Example 3.6.** *In [9], Johnstone constructed the first non-sober dcpo as  $X = \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$  with partial order defined by*

$$(m, n) \leq (m', n') \Leftrightarrow \text{either } m = m' \text{ and } n \leq n' \text{ or } n' = \infty \text{ and } n \leq m'.$$

*Then*

- (a)  $(X, \leq)$  is a dcpo,  $X$  is irreducible and  $X \neq cl(\{x\})$  for any  $x \in X$ .
  - (b) If  $F$  is a proper irreducible Scott closed set of  $X$ , then  $F = \downarrow(m, n)$  for some  $(m, n) \in X$ .
  - (c) If  $m \neq \infty$ ,  $\downarrow(m, n)$  is a down-linear element of  $Irr_\sigma(X)$ . If  $m = \infty$ , then  $\downarrow(m, n)$  is the supremum of the chain  $\{\downarrow(m, k) : k \neq \infty\}$  whose members are down-linear.
- Hence by Theorem 3.5, we deduce that dcpo  $X = \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$  is SCL-faithful.  
Thus an SCL-faithful dcpo need not be sober.

Next, we give another class of SCL-faithful dcpos.

**Remark 3.7.** (cf. [11]) Let  $A$  be a nonempty Scott closed set of a dcpo  $P$ . Then

- (i)  $A$  is a dcpo.
- (ii) For any subset  $B \subseteq A$ ,  $B$  is a Scott closed set of dcpo  $A$  iff it is a Scott closed set of  $P$ . Thus  $C_\sigma(A) = \downarrow_{C_\sigma(P)} A = \{B \in C_\sigma(P) : B \subseteq A\}$ .

A finite subset  $F$  of a dcpo  $P$  is way-below an element  $a \in P$ , denoted by  $F \ll a$ , if for any directed subset  $D \subseteq P$ ,  $a \leq \bigvee D$  implies  $D \cap \uparrow F \neq \emptyset$ . A dcpo  $P$  is quasicontinuous if for any  $x \in P$ , the family

$$fin(x) = \{F : F \text{ is finite and } F \ll x\}$$

is a directed family (for any  $F_1, F_2 \in fin(x)$  there is  $F \in fin(x)$  such that  $F \subseteq F_1 \cap \uparrow F_2$ ) and for any  $y \not\leq x$  there is  $F \in fin(x)$  satisfying  $y \not\in \uparrow F$  (see Definition III-3.2 of [3]). Every continuous dcpo is quasicontinuous.

Every quasicontinuous dcpo is sober (Proposition III-3.7 of [3]). A dcpo  $P$  is quasicontinuous iff the Scott open set lattice of  $P$  is hypercontinuous (Theorem VII-3.9 of [3]). From this and Remark 4, we have the following.

**Lemma 3.8.** Every quasicontinuous dcpo is SCL-faithful.

An element  $x$  of a dcpo  $P$  is called a quasicontinuous element if the sub-dcpo  $\downarrow x$  is a quasicontinuous dcpo.

**Theorem 3.9.** Let  $P$  be a dcpo. Then  $P$  is SCL-faithful if it satisfies the following two conditions:

- (1)  $\Sigma P$  is bounded sober;
- (2) every element of  $P$  is the supremum of a directed set of quasicontinuous elements.

*Proof.* Assume that  $P$  is a dcpo satisfying the two conditions. Let  $Q$  be a dcpo and  $F : C_\sigma(P) \rightarrow C_\sigma(Q)$  be an isomorphism. Then  $F$  restricts to an isomorphism  $F : Irr_\sigma(P) \rightarrow Irr_\sigma(Q)$ .

(1) Let  $x \in P$  be a quasicontinuous element. Then  $F(\downarrow x)$  is in  $C_\sigma(Q)$  and  $\downarrow_{C_\sigma(P)}(\downarrow x) = \{B \in C_\sigma(P) : B \subseteq \downarrow x\}$  is isomorphic via  $F$  to  $\downarrow_{C_\sigma(Q)} F(\downarrow x) = \{E \in C_\sigma(Q) : E \subseteq F(\downarrow x)\} = C_\sigma(F(\downarrow x))$  (all Scott closed sets of  $F(\downarrow x)$ ). Since  $\downarrow x$  is quasicontinuous, it is SCL-faithful. Hence  $\downarrow x$  is isomorphic to  $F(\downarrow x)$ , implying that there is a largest element in  $F(\downarrow x)$ , denoted by  $f(x)$ . It is easily observable that the mapping  $f$  is well defined on the set of quasicontinuous elements of  $P$ , and for any two quasicontinuous elements  $x_1, x_2 \in P$ ,  $f(x_1) \leq f(x_2)$  iff  $x_1 \leq x_2$ .

(2) If  $x \in P$  is the supremum of a directed set  $\{x_i : i \in I\}$  of quasicontinuous elements  $x_i$ , then  $F(\downarrow x) = \sup_{Irr_\sigma(Q)} \{F(\downarrow x_i) : i \in I\} = \sup_{Irr_\sigma(Q)} \{\downarrow f(x_i) : i \in I\} = \downarrow y_x$ , where  $y_x = \sup_Q \{f(x_i) : i \in I\}$  and  $f(x_i)$  is the element in  $Q$  defined for quasicontinuous elements  $x_i$  in (1). Let  $f(x) = y_x$  again.

Thus we have an monotone mapping  $f : P \longrightarrow Q$ . Following that  $F$  is an isomorphism, we have that  $f(x_1) \geq f(x_2)$  iff  $x_1 \geq x_2$ .

It remains to show that  $f$  is surjective.

(3) If  $y \in \downarrow f(P)$ , then  $\downarrow y \subseteq F(\downarrow x)$  for some  $x \in P$ . Since  $F$  restricts to an isomorphism between the dcpos  $\text{Irr}_\sigma(P)$  and  $\text{Irr}_\sigma(Q)$ , there is  $H \in \text{Irr}_\sigma(P)$  such that  $H \subseteq \downarrow x$  and  $F(H) = \downarrow y$ . But  $P$  is bounded-sober, so  $H = \downarrow x'$  for some  $x' \in P$ . It follows that  $y = f(x')$ , implying  $y \in f(P)$ . Therefore  $f(P)$  is a down set of  $Q$ . Also clearly  $f(P)$  is closed under sups of directed sets, so it is a Scott closed subset of  $Q$ .

(4) Since  $F$  is an isomorphism between the lattices  $C_\sigma(P)$  and  $C_\sigma(Q)$ ,  $Q = F(P) = F(\sup_{C_\sigma(P)} \{\downarrow x : x \in P\}) = \sup_{C_\sigma(Q)} \{F(\downarrow x) : x \in P\} = \sup_{C_\sigma(Q)} \{\downarrow f(x) : x \in P\}$ .

For each  $x \in P$ ,  $\downarrow f(x) \subseteq f(P)$  and  $f(P)$  is a Scott closed set of  $Q$ , it holds then that  $\sup_{C_\sigma(Q)} \{\downarrow f(x) : x \in P\} \subseteq f(P)$ . Therefore  $Q \subseteq f(P)$ , which implies  $Q = f(P)$ . Hence  $f$  is also surjective. The proof is thus completed.  $\square$

If  $x \in P$  is a down-linear element of a dcpo  $P$ , then  $\downarrow x$  is a chain, so it is continuous (hence quasicontinuous).

**Corollary 3.10.** *If  $P$  is a dcpo satisfying the following conditions, then  $P$  is SCL-faithful:*

- (1)  $P$  is bounded-sober.
- (2) every element  $a \in P$  is the supremum of a directed set of down-linear elements.

**Example 3.11.** *In order to answer the question whether every well-filtered dcpo is sober posed by Heckmann[5], Kou [10] constructed another non-sober dcpo  $P$  as follows:*

*Let  $X = \{x \in \mathbb{R} : 0 < x \leq 1\}$ ,  $P_0 = \{(k, a, b) \in \mathbb{R} : 0 < k < 1, 0 < b \leq a \leq 1\}$  and*

$$P = X \cup P_0.$$

*Define the partial order  $\sqsubseteq$  on  $P$  as follows:*

- (i) for  $x_1, x_2 \in X$ ,  $x_1 \sqsubseteq x_2$  iff  $x_1 = x_2$ ;
- (ii)  $(k_1, a_1, b_1) \sqsubseteq (k_2, a_2, b_2)$  iff  $k_1 \leq k_2, a_1 = a_2$  and  $b_1 = b_2$ .
- (iii)  $(k, a, b) \sqsubseteq x$  iff  $a = x$  or  $kb \leq x < b$ .

*If  $u = (h, a, b) \in P_0$ , then  $\downarrow u = \{(k, a, b) : k \leq h\}$  is a chain. If  $u = x \in P_0$ , then  $u = \bigvee \{(k, x, x) : 0 < k < 1\}$ , where each  $(k, x, x)$  is a down-linear element and  $\{(k, x, x) : 0 < k < 1\}$  is a chain. Thus  $P$  satisfies (2) of Corollary 3.10.*

*Let  $F$  be an irreducible nonempty Scott closed set of  $P$  with an upper bound  $v$ . If  $v = (h, a, b) \in P_0$ , then  $F \subseteq \downarrow (h, a, b) = \{(k, a, b) : k \leq h\}$ . Take  $m = \bigvee \{k : (k, a, b) \in F\}$ . Then  $F = \downarrow (m, a, b)$ , is the closure of point  $(m, a, b)$ .*

*Now assume that  $F$  does not have an upper bound in  $P_0$ , then  $v = x$  for some  $x \in P_0$ . If  $v \notin F$ , then due to the irreducibility of  $F$ , there exist  $a, b$  such that  $F \subseteq \{(k, a, b) : 0 < k < 1\}$ , which will imply that  $F$  has an upper bound of the form  $(m, a, b)$ , contradicting the assumption. Therefore  $v \in F$ , implying that  $F = \downarrow v$  (note that  $F = \downarrow F$  is a lower set) is the closure of point  $v$ .*

*It thus follows that  $P$  satisfies (1) as well. By Corollary 3.10,  $P$  is SCL-faithful.*

In [7], Ho and Zhao introduced the following notions.

**Definition 3.12.** *Let  $L$  be a poset and  $x, y \in L$ . The element  $x$  is beneath  $y$ , denoted by  $x \prec y$ , if for every nonempty Scott-closed set  $S \subseteq L$  with  $\bigvee S$  existing,  $y \leq \bigvee S$  implies  $x \in S$ . An element  $x$  of  $L$  is called C-compact if  $x \prec x$ . Let  $\kappa(L)$  denote the set of all the C-compact elements of  $L$ .*

Let  $P$  be a poset,  $A \subseteq P$  finite. The set  $mub(A)$  of the minimal upper bounds of  $A$  is complete, if for any upper bound  $x$  of  $A$ , there exists  $y \in mub(A)$  such that  $y \leq x$ .

A poset  $P$  is said to satisfy property  $m$ , if for all finite set  $A \subseteq P$ ,  $mub(A)$  is complete.

A poset  $P$  is said to satisfy property  $M$ , if  $P$  satisfies property  $m$  and for all finite set  $A \subseteq P$ ,  $mub(A)$  is finite.

**Remark 3.13.** Let  $L$  be a complete lattice and  $a \in L$  be a  $C$ -compact element. If  $x, y \in L$  such that  $a \leq x \vee y$ , then  $a \leq \bigvee(\downarrow x \cup \downarrow y)$  and  $\downarrow x \cup \downarrow y$  is Scott closed, so  $a \in \downarrow x \cup \downarrow y$ , implying  $a \leq x$  or  $a \leq y$ . Thus  $a$  is  $\vee$ -irreducible.

**Corollary 3.14.** For any dcpo  $P$ ,  $\kappa(C_\sigma(P)) \subseteq Irr_\sigma(P)$ . That is  $C$ -compact closet sets are all irreducible.

**Lemma 3.15.** [4] Let  $P$  be a dcpo. Then

- (1) For all  $x \in P$ ,  $\downarrow x \in \kappa(C_\sigma(P))$ .
- (2) If  $P$  satisfies property  $M$ , then  $A \in \kappa(C_\sigma(P))$  iff  $A = \downarrow x$  for some  $x \in P$ .

The following theorem gives the third class of SCL-faithful dcpos using property  $M$ .

**Theorem 3.16.** If  $P$  is a dcpo satisfying property  $M$  and the condition (2) in Theorem 3.9, then  $P$  is SCL-faithful

*Proof.* Let  $P$  be a dcpo satisfying condition (2) in Theorem 3.9 and property  $M$ , and  $Q$  be a dcpo with an order isomorphism  $F : C_\sigma(P) \rightarrow C_\sigma(Q)$ .

Then the restrictions  $F : \kappa(C_\sigma(P)) \rightarrow \kappa(C_\sigma(Q))$  and  $F : Irr_\sigma(P) \rightarrow Irr_\sigma(Q)$  are all order isomorphisms.

For each  $q \in Q$ , by Lemma 3.15(1),  $\downarrow q \in \kappa(C_\sigma(Q))$ , then  $F^{-1}(\downarrow q) = \downarrow x_q$  for a unique  $x \in P$  by Lemma 3.15(2). Now define a map  $g : Q \rightarrow P$  such that  $g(q) = x_q$  iff  $F^{-1}(\downarrow q) = \downarrow x_q$ . The mapping  $g$  satisfies the condition that  $g(q_1) \leq g(q_2)$  iff  $q_1 \leq q_2$  since  $F^{-1}$  is an isomorphism. Note that  $\kappa(C_\sigma(Q)) \cong \kappa(C_\sigma(P)) \cong P$  is a dcpo.

Since  $P$  satisfies the condition (2) in Theorem 3.9, by the proof of Theorem 3.9 there is a monotone mapping  $f : P \rightarrow Q$  such that  $F(\downarrow x) = \downarrow f(x)$  holds for every  $x \in P$  (note that parts (1) and (2) of proof of Theorem 3.9 do not need the condition that  $P$  is bounded sober).

Then for any  $x \in P$ ,  $\downarrow x = F^{-1}(\downarrow f(x))$ , so  $x = g(f(x))$ . Thus  $g : Q \rightarrow P$  is also a surjective mapping, therefore an isomorphism between  $P$  and  $Q$ , as desired.  $\square$

Note that Kou's and Johnstone's examples of dcpos are bounded sober but do not have property  $M$ .

#### 4. REMARKS AND SOME POSSIBLE FURTHER WORK

We close the paper with some extra remarks and problems for further exploration.

**Remark 4.1.** (1) Recently, Ho, Jung and Xi [6] constructed a pair of non-isomorphic dcpos having isomorphic Scott topologies, showing the existence of non-SCL-faithful dcpos. Their counterexample also reveals that sobriety is not a sufficient condition for a dcpo to be SCL-faithful.

(2) If  $P$  is an SCL-faithful dcpo and  $P^*$  is the dcpo obtained by adding a top element to  $P$ , then one can show that  $P^*$  is also SCL-faithful. Let  $X$  be the dcpo of Johnstone. Then  $X^*$  is SCL-faithful, but  $X^*$  is not bounded sober ( $X$  is an irreducible Scott closed set of  $X^*$

which is not the closure of any point of  $X^*$ ). Thus a SCL-faithful dcpo need not be bounded sober. So, bounded sobriety is not a necessary condition for a dcpo to be SCL-faithful.

(3) The bounded sobriety in Theorem 3.9 might be further weakened. We can try other conditions which are weaker than this, like the “dominatedness” used in [6].

(4) Given a class  $\mathcal{M}$  of dcpos, define

$$\mathcal{M}^b = \{P : P \text{ is a dcpo and for any } Q \in \mathcal{M}, C_\sigma(P) \cong C_\sigma(Q) \text{ implies } P \cong Q\}.$$

A class  $\mathcal{M}$  of dcpos is called reflexive if  $\mathcal{M}^{bb} = \mathcal{M}$ .

The class  $\mathcal{S}$  of all SCL-faithful dcpos and the class DCPO of all dcpos are reflexive. Do we have other reflexive classes of dcpos other than these two?

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